Journal of Applied Mathematics and Mechanics 72 (2008) 26–32

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Optimal body design using localized interaction models☆

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ARTICLE INFO

Article history:
Received 19 April 2007

ABSTRACT

The problem of the designing bodies of minimum drag for a specified area of the base and a specified area of the ("windward") surface around which the flow occurs is considered in the approximation of an arbitrary localized interaction model. New necessary conditions for minimum drag are obtained which are stronger than the Legendre condition. It is shown that, in the approximation used, the optimal configurations in the general case contain end faces and cylindrical segments of the boundary extremum, which appear due to the existence of limits of applicability of local models. It is established that the solutions previously obtained are incomplete. A complete solution of the problem is constructed.

1. Formulation of the problem

In the fairly arbitrary localized interaction model (LIM), a force[1]

$$\mathbf{D} = \iint\limits_S \{[p_\infty + qc_p(\alpha)]\mathbf{n} + qc_f(\alpha)\mathbf{t}\}dS$$

$$\alpha = \alpha(y,z) \equiv \mathbf{n}\cdot\mathbf{k} = \frac{1}{\sqrt{1+x_z^2+x_y^2}},$$

$$\mathbf{t}\cdot[\mathbf{n}\times\mathbf{k}] = 0, \quad \mathbf{t}\cdot\mathbf{k} = \sqrt{1-\alpha^2} \tag{1.1}$$

acts on the ("windward") surface (WS) past which the flow occurs. Here and in Fig. 1, *a*, the unit vector $\mathbf{k}$ of the x axis of the Cartesian coordinates xyz is directed along the free stream velocity vector $\mathbf{V}_\infty$, the subscript ∞ is ascribed to the free stream parameters, $x = x(z, y)$ is the equation of the "windward" surface (WS), p and ρ are the pressure and density of the gas, $V = |\mathbf{V}|$ and $q = \rho_\infty V_\infty^2$, S_b is the area of the base of the body which is bounded by the curve Γ, S is the area of the WS and $\mathbf{n}$ is its inward normal. On the WS

$$0 \le \alpha \le 1 \tag{1.2}$$

In the LIM, the coefficients of the pressure $c_p > 0$ and the friction $c_f \ge 0$ acting on an element of the surface S are known functions solely of α and the free stream parameters ($c_f = 0$ is written in order to include Newton's formula in the treatment). The friction force is directed along the vector $\mathbf{t}$, tangential to the surface past which the flow occurs. By assumption, the vectors $\mathbf{n}$, $\mathbf{k}$ and $\mathbf{t}$ are coplanar.

We shall confine ourselves to bodies which have a base lying in the yz plane (when $x = 0$) and the projection of the WS onto this plane coincides with it. If S_b is adopted as the scale of area, then $S \ge S_b = 1$. By condition (1.2), together with the sloping parts where $0 < \alpha < 1$, the windward surface (WS) can contain faces and cylindrical segments for which α takes its limit admissible values: $\alpha = 1$ and $\alpha = 0$. If S_1 and S_0 are their areas, then, when account is taken of expression (1.1), the drag coefficient of the WS χ and its area are given by the equalities

$$\chi = \frac{\mathbf{D}\cdot\mathbf{k} - S_b p_\infty}{S_b q} = \iint\limits_{S_b \backslash S_1} F(\alpha)dS_b + c_{f0}S_0 + c_{p1}S_1,$$

$$S = \iint\limits_{S_b \backslash S_1} \frac{dS_b}{\alpha} + S_0 + S_1$$

$$F(\alpha) = c_p(\alpha) + \frac{c_f(\alpha)}{\alpha}\sqrt{1-\alpha^2},\ c_{f0} = c_f(0),\ c_{p1} = c_p(1),$$

$$0 \le S_0 < S,\ S \ge S_b = 1,\ 0 \le S_1 \le 1 \tag{1.3}$$

Here and henceforth $S_b \backslash S_1$ denotes integration over the base of the body without the projection onto it of the end face (or faces) of area S_1.

The problem of designing the WS which gives the minimum of χ for a specified area S was solved earlier[2] in the case of the LIM

$$c_p(\alpha) = \alpha^2, \quad c_f(\alpha) = c_f = \text{const} \tag{1.4}$$

which corresponds to Newton's formula and a constant friction coefficient c_f. This problem has been considered in Refs. 3 and 4 when $c_p(\alpha) = \alpha^2$ and $c_f = 0$. When solving the same problem in Ref. 5, the local "drag law", that is, the form of the

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 1, pp. 41–53, 2008.

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doi:10.1016/j.jappmathmech.2008.03.008

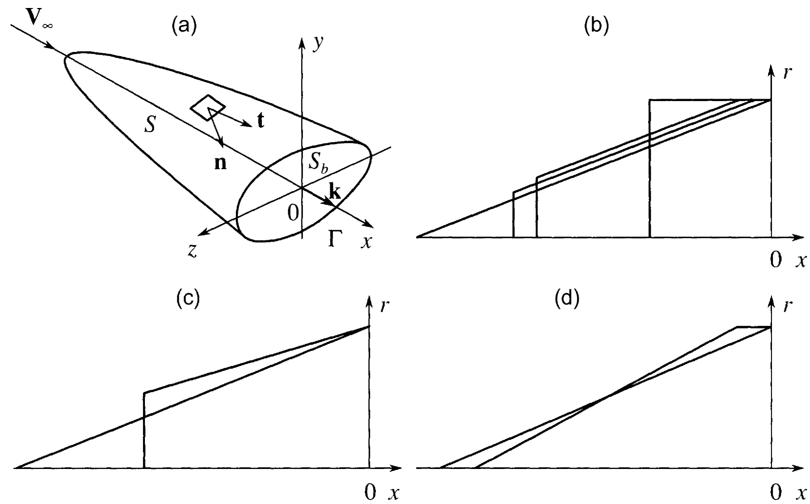

Fig. 1.

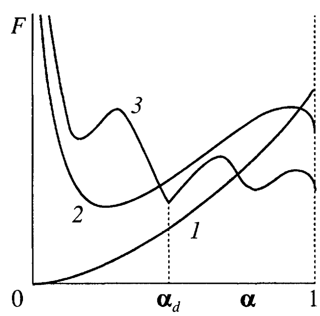

Fig. 2.

coefficients $c_p(\alpha)$ and $c_f(\alpha)$, was not specified. The possibility of non-zero values of S_0 and S_1 was not considered in Refs. 2–5.

As is shown below, the form of the relation $F(\alpha)$ when $0 \le \alpha \le 1$ is important a the complete solution of the problem. Three curves, giving different forms of $F(\alpha)$ are presented in Fig. 2. Curve *1*, which corresponds to Newton's formula with a zero friction coefficient, has a single minimum which is simultaneously also a boundary minimum (when $\alpha = 0$ at the left-hand boundary of the admissible range of variation of α) and a classical minimum $F_\alpha = dF/d\alpha = 2\alpha = 0$ when $\alpha = 0$. In the case of curve *2*, as in the LIM (1.4), one minimum is classical and the second is a "boundary" minimum (when $\alpha = 1$). Curve *3*, together with a single boundary and two classical minima of $F(\alpha)$, also has a further minimum at the point of discontinuity. A discontinuity and even a break in the curve $F = F(\alpha)$ are possible if different drag laws are used in the LIM: one for smaller and one for larger values of α. In the case of curves *2* and *3*, $F(0) = \infty$.

2. Necessary optimality conditions when there are no cylindrical or end face segments

In order to obtain the necessary conditions for a minimum of χ for a specified area S, the Lagrange functional $I = \chi + \lambda S$ is combined with a constant undetermined multiplier λ. Since S is fixed, the conditions for χ to be a minimum are identical to those for I to be a minimum.

If, as was done previously in Refs. 2–5, cylindrical and end face segments are not provided for in the optimal WS and the treatment is restricted to a smooth function $F(\alpha)$, then, for the increment $\Delta\chi = \Delta I$, we obtain

$$\Delta\chi = \iint\limits_{S_b} \left[J_\alpha \delta\alpha + \frac{J_{\alpha\alpha}}{2}(\delta\alpha)^2 + \ldots \right] dS_b =$$

$$\iint\limits_{S_b} \left\{ -\left[\frac{\partial(J_\alpha \alpha_{x_y})}{\partial y} + \frac{\partial(J_\alpha \alpha_{x_z})}{\partial z} \right] \delta x + \frac{J_{\alpha\alpha}}{2}(\delta\alpha)^2 + \ldots \right\} dS_b$$

$$J = J(\alpha, \lambda) = F(\alpha) + \frac{\lambda}{\alpha} = c_p(\alpha) + \frac{c_f(\alpha)}{\alpha}\sqrt{1-\alpha^2} + \frac{\lambda}{\alpha} \tag{2.1}$$

In order to obtain the second expression for $\Delta\chi$, account has been taken of the fact that, on the curve Γ, the variation $\delta x = 0$, since the base of the body belongs to the $x = 0$ plane.

According to the first expression for $\Delta\chi$, the necessary conditions for a minimum of χ reduce to

$$J_\alpha(\alpha, \lambda) = 0, \quad J_{\alpha\alpha}(\alpha, \lambda) \ge 0 \tag{2.2}$$

These conditions were obtained for the first time in Ref. 2. It is seen from their derivation that the specific form of the functions $c_p(\alpha)$ and $c_f(\alpha)$ used here is of no significance. It is only essential that the integrand in the Lagrange functional is individually independent of x_y and x_z and is given in terms of a function α which itself represents a certain combination of these derivatives. Moreover, for conditions (2.2) to hold, its form is also unimportant. For such problems, the equality (2.2) is Euler's equation (as it was called in Ref. 2) and the inequality is the Legendre condition.

By virtue of the equality (2.2), the value of α is constant for the optimal WS. According to the expression for S (1.3) when $S_0 = S_1 = 0$, it is equal to

$$\alpha = \alpha_m \equiv 1/S = S_b/S \le 1 \tag{2.3}$$

Substituting of α_m into the formula for χ (1.3) with $S_0 = S_1 = 0$ we obtain the drag of the optimal WS

$$\chi_m = F(1/S) \tag{2.4}$$

The Lagrange multiplier λ, defined by the equality (2.2), is only required in this case in order to check the Legendre condition which, together with the expressions for λ, is written in the form

$$\lambda = \alpha_m^2 F_{\alpha_m}(\alpha_m), \quad [\alpha_m^2 F_{\alpha_m}]_{\alpha_m} \ge 0 \tag{2.5}$$

The optimal value α_m, defined by equality (2.3), is only identical with the coordinates of the minimum of the function $F(\alpha)$ for completely fixed values of S. So, in the case of curves *2* and *3* in Fig. 2, these will be the points of a boundary minimum when $S = 1$. In the case of the LIM corresponding to curve *3*, the lowest drag is obtained at the point of discontinuity $\alpha = \alpha_d$. In such "irregular" cases, the derivative of F_α is non-zero at the corresponding points and, at the point of discontinuity of curve *3*, it breaks. In spite of this, formulae (2.3) and (2.4) hold as before. For example, in the case of an end face of the unique WS which is possible when $S = 1$, $\alpha_m = 1$ and $\chi_m = F(1)$ according to these formulae.

Within the framework of the LIM (1.4), the Legendre condition reduces to the inequality

$$\varphi_L(\alpha_m, c_f) \equiv 6 - \frac{c_f}{\alpha_m(1-\alpha_m^2)^{3/2}} = 6 - \frac{c_f S^4}{(S^2-1)^{3/2}} \geq 0 \tag{2.6}$$

which is equivalent to condition (83) in Ref. 2.

According to formulae (1.1) and (2.3), the equation for α, which defines the optimal WS, has the form

$$x_z^2 + x_y^2 = \alpha_m^{-2} - 1 = S^2 - 1 \geq 0 \tag{2.7}$$

In the case of an arbitrary shape of the base, this first order partial differential equation is solved by the method of "characteristic strips".[6,7] By virtue of Eq. (2.7), the resulting form of the optimal WS is completely determined by the magnitude of S, that is, by the ratio of the specified area of the WS to the specified area of the base, and is absolutely independent of the drag law. In particular, if the base is a circle, then the WS based on it is the surface of a circular cone.

The fact that the optimal form is independent of the drag law is already strange because the Legendre condition, which depends both on S and on the drag law, can be violated. For example, within the framework of the LIM (1.4), the modulus of the negative second term in (2.6) tends to infinity when $S \to 1$ and, for large S, increases linearly with S. Consequently, starting from a certain S, the inequality (2.6) is necessarily violated. However, the inevitability of the violation of the Legendre condition at large values of S and values of S close to unity was not remarked upon in Ref. 2.

The Bunimovich and Dubinskii,[5] while aspiring to solve the problem in the case of an arbitrary LIM, wrote out the Euler equation in the "traditional" form

$$\frac{\partial(J_\alpha \alpha_{x_y})}{\partial y} + \frac{\partial(J_\alpha \alpha_{x_z})}{\partial z} = 0 \tag{2.8}$$

which corresponds to the second expression for the increment in χ in (2.1), as the necessary conditions for a minimum of χ.

Equation (2.8), which is a second order equation in x, must be solved with the boundary condition $x|_\Gamma = 0$. It is incomparably more complex than Eq. (2.7) that is a corollary of Eq. (2.2) which, in the case of this problem, is also the initial Euler equation. If J does not contain x, then the transition from the first way of writing the expression for $\Delta\chi$ in (2.1) to the second way of writing this expression and to Eq. (2.8), which was referred to as the "Euler-Ostrogradskii equation" in Ref. 5, is only necessary if J is depends both on x_y and x_z and not on their combination as in a LIM. Moreover, it is even harmful in this problem. Actually, by virtue of the first condition of (2.2), $J_\alpha = 0$ which, with the equalities which follow from this and the inequalities (2.3)–(2.5), gives a unique minimum of χ. Unlike this, $J_\alpha = 0$ is just one of the solutions of Eq. (2.8), and it is necessary to prove that there are no other, possibly better, solutions of Eq. (2.8). However, this question was not discussed in Ref. 5. Finally, as shown at the end of Section 3, the value of J_α is non-zero in the case of bodies with $\alpha = \alpha_d$, which corresponds to the discontinuity in curve *3* in Fig. 2.

3. The necessary conditions of optimality when cylindrical and end face segments are admitted

Even in the case of the extremely simple LIM (1.4), the solution, obtained earlier in Ref. 2 and described in Section 2, is incomplete on account of the violation of the Legendre condition (2.6). Accordingly, we admit the existence in the optimal WS of cylindrical parts (based on the boundary of the base Γ, for example) and windward ("leading") end face segments that, in the formulation that has been adopted, can appear as segments of a boundary extremum (SBE). For the first of these $\alpha = 0$ and the admitted variations $\delta\alpha \geq 0$ and, for the second, $\alpha = 1$ and the admitted variations $\delta\alpha \leq 0$. With the assumption of SBE

$$\begin{aligned}\Delta\chi &= \iint\limits_{S_b \setminus S_1} \delta J(\alpha,\lambda)dS_b + \oint\limits_{\Gamma_1}[c_{p1} + \lambda - J(\alpha,\lambda)]\delta n^1 d\Gamma_1 \\ &\quad + (c_{f0} + \lambda)\Delta S_0 = \\ &= \iint\limits_{S_b \setminus S_1}\Big[J_\alpha(\alpha,\lambda)\delta\alpha + \frac{J_{\alpha\alpha}(\alpha,\lambda)}{2}(\delta\alpha)^2 + \ldots\Big]dS_b + \ldots\end{aligned} \tag{3.1}$$

Here, the contour integral is chosen in the plane $x = \text{const}$ along the boundary Γ_1 which separates the end face (when $S_1 > 0$) from the inclined part of the WS (the segment of the two-sided extremum (STE)), δn^1 is the displacement of Γ_1 along the outward normal to itself (in the same plane) and $J(\alpha, \lambda)$ is the value of J in Γ_1 as viewed from the STE. The first term in the second expression for $\Delta\chi$ is written for the case when the function $J(\alpha, \lambda)$ is smooth in the neighbourhood of the variable value of α.

In cases, when the second way of writing the first term in formulae (3.1) is suitable, we start to obtain the necessary conditions for a minimum from the case considered in Section 2 when there are no segments of a boundary extremum (SBE's) ($S_1 = S_0 = 0$). Due to the choice of the multiplier λ, at a certain "compensating" point k, we put

$$[J_\alpha(\alpha,\lambda)]_k \equiv [F_\alpha(\alpha) - \lambda\alpha^{-2}]_k = 0 \tag{3.2}$$

and, simultaneously with the variation of α at an arbitrary point of the WS, we preserve S at the price of a change in α in the neighbourhood of the point k while keeping S_b unchanged. Due to this, all the variations and increments in (3.1) can be considered as being independent.

On the sloping part of the WS, the variations $\delta\alpha$ are arbitrary and, therefore, the necessary condition for a minimum of χ, determining the optimal shape of the WS, reduces in the first place to equality (2.2), that is, equality (3.2) is not only satisfied at the point k but for the whole of the required WS. Consequently, formulae (2.3)–(2.5) hold as before for the constant optimal $\alpha = \alpha_m$, the coefficient χ_m and the Lagrange multiplier λ. After this, in the case being considered, expression (3.1) for $\Delta\chi$ takes the form

$$\begin{aligned}\Delta\chi &= \frac{1}{2}\iint\limits_{S_b \setminus S_1} J_{\alpha\alpha}(\alpha,\lambda)(\delta\alpha)^2 dS_b \\ &\quad + [c_{p1} + \lambda - J(\alpha,\lambda)]\Delta S_1 + (c_{f0} + \lambda)\Delta S_0\end{aligned} \tag{3.3}$$

From here, as previously, the Legendre condition is obtained, which reduces to the inequality (2.5) but not only. If the optimal WS does not contain cylindrical and end face parts ($S_0 = S_1 = 0$), then their introduction ($\Delta S_0 > 0$, $\Delta S_1 > 0$) at any point of such a surface must not reduce the drag. In accordance with this, such WSs must satisfy two further necessary conditions for optimality. After eliminating of the multiplier λ, they take the form

$$c_{f0} + \alpha_m^2 F_{\alpha_m}(\alpha_m) \geq 0,$$
$$c_{p1} + (\alpha_m - 1)\alpha_m F_{\alpha_m}(\alpha_m) - F(\alpha_m) \geq 0 \tag{3.4}$$

Within the framework of the LIM (1.4), these conditions reduce to

$$\varphi_0(\alpha_m, c_f) \equiv 2\alpha_m^3 - c_f\frac{1-\sqrt{1-\alpha_m^2}}{\sqrt{1-\alpha_m^2}} = \frac{2}{S^3} - c_f\frac{S-\sqrt{S^2-1}}{\sqrt{S^2-1}} \geq 0$$
$$\varphi_1(\alpha_m, c_f) \equiv 1 + \alpha_m - 2\alpha_m^2 - \frac{c_f}{\sqrt{1-\alpha_m^2}} = \frac{S^2+S-2}{S^2} - \frac{c_f S}{\sqrt{S^2-1}} \geq 0 \tag{3.5}$$

The expressions in terms of S are obtained after the replacement of α_m in accordance with formula (2.3).

The functions $\varphi_{0,1}(\alpha_m, c_f)$ in inequalities (3.5) as well as the function $\varphi_L(\alpha_m, c_f)$ in the Legendre condition (2.6) are of alternating sign when $c_f > 0$ and they are negative for any $0 \leq \alpha_m \leq 1$ when a certain value of the friction coefficient c_f is exceeded. The relations $c_f(\alpha_m)$ are shown in Fig. 3 in which the curves *0*, *1* and *L* correspond to the functions φ_0, φ_1 and φ_L vanishing. Over the whole possible range of variation of α_m, the curve *L* is above at least one of the curves: *0* or *1*. Hence, for the LIM (1.4) with a non-zero friction coefficient, the constraints on c_f following from inequalities (3.5) are stronger than those imposed by the Legendre condition. For all values of α_m which are admissible for the WS, inequalities (2.6) and (3.5) are only satisfied for a LIM with zero friction. Consequently, it is only in this case, considered earlier in Refs. 3 and 4 that the solution described in Section 2 holds for all $S \geq 1$.

When at least one of inequalities (2.5) and (3.5) is violated (for the LIM (1.4), (2.6) and (3.5)), one must allow the existence of SBE of one or both types, which have been considered, in addition to a sloping STE in the unknown WS. In particular, the WS can contain STE in which condition (2.2) holds as before and both of the SBE are of non-zero area ($S_1 > 0$ and $S_0 > 0$). In this case, the constancy of $\alpha = \alpha_m$ with $0 < \alpha_m < 1$ follows, as before, from the Euler equation, that is, equality (2.2), which holds for the STE. Now, however, instead of formula (2.3) for α_m, the relation

$$S = \frac{1-S_1}{\alpha_m} + S_1 + S_0 \tag{3.6}$$

obtained from the second equality of (1.3), holds.

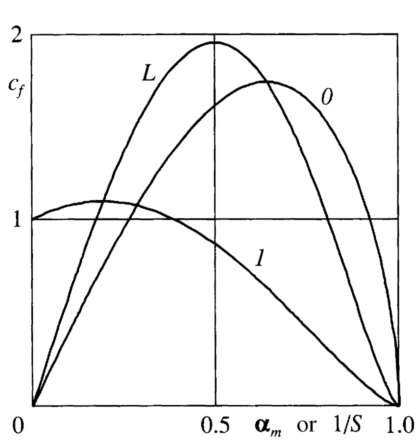

Fig. 3.

When $S_1 > 0$ and $S_0 > 0$, the increments ΔS_1 and ΔS_0, unlike in the case considered above, are arbitrary, and, from expression (3.1) for $\Delta\chi$, we therefore obtain the equalities ("transversality conditions")

$$c_{f0} + \lambda = 0 \tag{3.7}$$
$$c_{p1} - c_p(\alpha_m) - \frac{c_f(\alpha_m)}{\alpha_m}\sqrt{1-\alpha_m^2} - \frac{1-\alpha_m}{\alpha_m}\lambda = 0 \tag{3.8}$$

instead of inequalities (3.4) as additional necessary conditions for an extremum of χ.

The two quantities α_m and λ, to be determined, occur in conditions (3.7) and (3.8) and the Euler equation (2.2). Eliminating λ using condition (3.7), we arrive at two independent equations for determining α_m

$$c_{p1} - c_{f0} - \Phi(\alpha_m) = 0, \quad \frac{d\Phi(\alpha_m)}{d\alpha_m} = 0$$
$$\Phi(\alpha_m) = F(\alpha_m) - \frac{c_{f0}}{\alpha_m} = c_p(\alpha_m) + \frac{c_f(\alpha_m)}{\alpha_m}\sqrt{1-\alpha_m^2} - \frac{c_{f0}}{\alpha_m} \tag{3.9}$$

Although it is highly unlikely that these equations are simultaneously satisfied, it is possible in principle. There is just a single equation (3.6) for determining S_0 and S_1 in these ("special") cases for a specified S and such α_m. As a result, a single-parameter family of optimal WSs is obtained. Substituting S_0 and c_{p1}, found from relation (3.6) and the first equality of (3.9), into formula (1.3) for χ, we obtain

$$\chi_m = \Phi(\alpha_m) + Sc_{f0} = c_{p1} + c_{f0}(S-1)$$

Hence, as might have been expected, the drag of all of the bodies of the above-mentioned single-parameter family, including a sharp cone and a blunt cylinder, is the same. Two such bodies in the case of a circular base, a blunt cylinder and a pointed cone from Section 2, are presented in Fig. 1, *b* (*r* is the radial coordinate).

In the case of a WS only with an end face SBE ($S_1 > 0$, $S_0 = 0$), the necessary conditions for a minimum of χ reduce to equalities (2.5) and (3.8) and to the inequality

$$c_{f0} + \lambda \geq 0$$

Substituting the value of λ, found from equality (3.8), into it, we obtain

$$c_{p1} - c_{f0} - \Phi(\alpha_m) \geq 0 \tag{3.10}$$

with the function $\Phi(\alpha_m)$ defined above. Substitution of the same value of λ into Euler's equation, that is, equality (2.5), gives an equation for determining α_m

$$c_{p1} - c_p(\alpha_m) - \frac{c_f(\alpha_m)}{\alpha_m}\sqrt{1-\alpha_m^2} - (1-\alpha_m)\alpha_m F_{\alpha_m}(\alpha_m) = 0 \tag{3.11}$$

If the Legendre condition (inequality (2.5)) and the inequality (3.10) are satisfied for value of α_m found from Eq. (3.11), then S_1 and χ_m are calculated using the formulae

$$S_1 = \frac{1-\alpha_m S}{1-\alpha_m}, \quad \chi_m = \frac{\alpha_m c_p(\alpha_m) + c_f(\alpha_m)\sqrt{1-\alpha_m^2}}{1-\alpha_m}(S-1) + c_{p1}\frac{1-\alpha_m S}{1-\alpha_m}$$

In the case of a circular base, the optimal truncated cone with end face SBE and the pseudo-optimal pointed cone are shown in Fig. 1, *c*.

In the case of the LIM (1.4), Eq. (3.11) and inequality (3.10) are written in the form

$$(1+\alpha_m-2\alpha_m^2)\sqrt{1-\alpha_m^2} = c_f,$$

$$\alpha_m-\alpha_m^3+c_f(1-\alpha_m-\sqrt{1-\alpha_m^2})\geq 0 \tag{3.12}$$

If the resulting expression for c_f is substituted into this inequality, it takes the form

$$2\alpha_m^3-(1+\alpha_m-2\alpha_m^2)(1-\sqrt{1-\alpha_m^2})\geq 0 \tag{3.13}$$

According to calculations, this inequality is satisfied when $\alpha_m \geq \alpha_{m*}$, where $\alpha_{m*}\approx 0.2866$ is the ordinate of the point of intersection of the curves 0 and 1 in Fig. 3. At this point, $c_{f0}=c_{f1}\equiv c_{f*}\approx 1.0752$.

The optimal WSs with cylindrical SBE ($S_0>0$, $S_1=0$) are treated in a similar manner. In this case, α_m is given by the equation

$$F_{\alpha_m}(\alpha_m)+\frac{c_{f0}}{\alpha_m^2}\equiv\frac{d\Phi(\alpha_m)}{d\alpha_m} = 0 \tag{3.14}$$

with the function $\Phi(\alpha_m)$, introduced by the last equality of (3.9). Here, the conditions, which have in the form of inequalities, reduce to the Legendre condition (inequality (2.5)) and to condition (3.10), that is, at first glance, they do not differ from those obtained for WSs with end face SBE. However, since the basic equation (3.14) defining α_m differs from equation (3.11), which plays a similar role in the case of WSs with end face SBE, the resulting inequality does differ. For instance, in the case of the LIM (1.4), the equation and inequalities (3.12) and (3.13) are replaced by

$$\frac{2\alpha_m^3\sqrt{1-\alpha_m^2}}{1-\sqrt{1-\alpha_m^2}} = c_f,\quad 1+\alpha_m-2\alpha_m^2-\frac{c_f}{\sqrt{1-\alpha_m^2}}\geq 0$$

$$(1+\alpha_m-2\alpha_m^2)(1-\sqrt{1-\alpha_m^2})-2\alpha_m^3\geq 0 \tag{3.15}$$

Unlike inequality (3.13), the last condition of (3.15) is satisfied when $\alpha_m\leq\alpha_{m*}$. When $\alpha_m\leq\alpha_{m*}$, the left-hand sides of inequalities (3.13) and (3.15) simultaneously vanish.

If the Legendre condition (inequality (2.5)) and inequality (3.10) are satisfied in the case of α_m found from Eq. (3.14), then S_0 and χ_m are calculated from the formulae

$$S_0 = \frac{\alpha_m S-1}{\alpha_m},$$

$$\chi_m = \frac{\alpha_m c_p(\alpha_m)+c_f(\alpha_m)\sqrt{1-\alpha_m^2}+(\alpha_m S-1)c_{f0}}{\alpha_m}$$

In the case of a circular base, the optimal WS is a combination of a cone with cylindrical SBE and the pseudo-optimal pointed cone shown in Fig. 1, *d*.

If the curve $F=F(\alpha)$ is not smooth with a minimum at the break (when $\alpha=\alpha_d$), as in the case of curve *3* in Fig. 2, it can be possible that $\alpha_m=\alpha_d$ for a certain value of *S*. In this case, the multiplier λ which causes the left-hand side of equality (3.2) to vanish at the "compensating" point *k* with $\alpha_k=\alpha_d$ is indeterminate at first glance. In fact, its value at the point *k* is chosen from equality (3.2) to be different depending on the sign of the variations $\delta\alpha$ at other points. If they are positive, then, in order to keep *S* constant in the neighbourhood of the point *k*, the variation $\delta\alpha$ must be negative. In such cases, by the meaning the compensating point, $\lambda=\alpha_d^2F_{\alpha d-}$. At other points, the coefficient of non-negative $\delta\alpha$ in the integrand in formula (3.1) is equal to

$$J_\alpha(\alpha,\lambda) = F_\alpha(\alpha)_{d+}-\frac{\lambda}{\alpha_d^2} = F_\alpha(\alpha)_{d+}-F_\alpha(\alpha)_{d-}>0$$

leading, as must be the case, to an increase in the drag. If, however, at points differing from the compensating point, $\delta\alpha\leq 0$, then

$$\lambda = \alpha_d^2F_\alpha(\alpha)_{d+},$$

$$J_\alpha(\alpha,\lambda) = F_\alpha(\alpha)_{d-}-\frac{\lambda}{\alpha_d^2} = F_\alpha(\alpha)_{d-}-F_\alpha(\alpha)_{d+}<0$$

Being equivalent to the preceding inequality, this inequality also leads to an increase in χ for the optimal WS. In similar situations, it is naturally not necessary for the Legendre condition to be satisfied.

4. The structure of the optimal surface in the general case

In the problem being considered, according to the conditions obtained above, the form of the optimal WS, notwithstanding the assertions in the Refs. 2 and 5, not only depends on the magnitude of *S* but, also, on the LIM. We will demonstrate this by taking LIM (1.4) as an example. We start with small friction coefficients. The horizontal $c_f=\text{const}\ll 1$ in Fig. 3 lies almost as a whole beneath the curves 0 and 1, which bound the domain from above in which inequalities (3.5), that is, the necessary conditions for a minimum in χ, are satisfied. Suppose $\alpha_{m0}=\alpha_m(c_f)$ for the increasing part of the curve 0, $\alpha_{m1}=\alpha_m(c_f)$ for the decreasing part of the curve *1* and that the specified magnitude of *S* and the friction coefficient $c_f<c_{f*}\approx 1.0752$ are such that $\alpha_{m0}\leq 1/S\leq\alpha_{m1}$. In such situations, the optimal $\alpha_m\leq 1/S$ and, in the case of a circular base, one of the optimal WSs is the surface of a circular cone. The semiangle at its vertex $\theta=\arcsin(1/S)$.

Together with a conical WS, it is possible to construct an infinite number of optimal spatial WSs which, in the case of those *S* based on a circular base, have the same drag $\chi_m=F(1/S)$. This also holds in the case of LIMs without friction which have been previously considered.[3,4] One of these spatial WSs, constructed using the method developed in Ref. 8, is shown in Fig. 4, *a* where a circular cone is also shown. According to the method of construction,[1,8] $\alpha=\alpha_m=1/S$ for all elements of the spatial WS. This, by virtue of formulae (1.3) written for $S_1=S_0=0$, also ensures the same values of χ and *S* in the case of all such bodies. The same method enables us

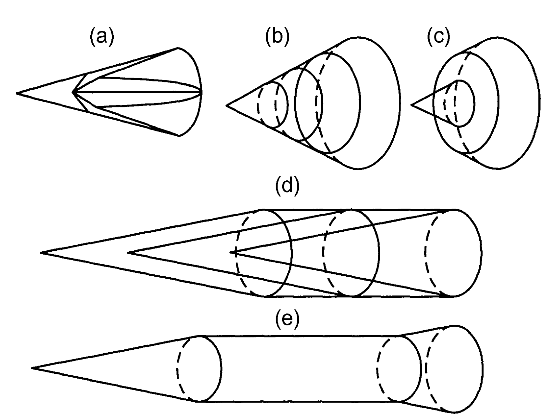

Fig. 4.

to construct an unbounded set of spatial bodies with bases which differ from a circular base. They all have the same drag for a fixed value of S.

Now, suppose that, when $c_f < c_{f*} \approx 1.0752$, the magnitude of S is such that $\alpha_{m1} < 1/S \leq 1$. In such situations, the optimal WS consists of an end face and STE for which $\alpha_m = \alpha_{m1}(c_f)$ for all S from the above-mentioned range, that is, it is constant. In the case of a circular base, one of the optimal WSs is the surface of a truncated circular cone. Three such truncated cones are shown in Fig. 4, *b*. When $S \to 1$, the height of the truncated zone tends to zero. On the other hand, as in the area of the WS increases when its value reaches $S = 1/\alpha_{m1}(c_f)$,, the cone becomes pointed. The semi-angle at the vertex of the truncated cone is $\theta = \arcsin \alpha_{m1}(c_f) \leq \arcsin(1/S)$ and the equality only holds in the limiting case when $S = 1/\alpha_{m1}(c_f)$. This fact is reflected in Fig. 1, *c*. On the other hand, it can be seen from Fig. 4, *b* that the angles at the vertex of such cones are greater than the angle of the pointed cone shown in Fig. 4, *a*. The surface of the truncated cone (Fig. 4, *b*) can be subdivided into several surfaces, for example, the two surfaces as shown in Fig. 4, *c*. Finally, the conical surfaces can be replaced by spatial surfaces constructed using the technique developed earlier in Ref. 8. In the case of a circular base, such WSs can be obtained by removing the head part of the body shown in Fig. 4, *a* with the plane $x = \text{const}$. The difference from the preceding situation lies solely in the fact that the corresponding spatial WS is constructed for $\alpha_m = \alpha_{m1}(c_f)$, rather than for $\alpha_m = 1/S$.

Suppose, as before, that $c_f \leq c_{f*} \approx 1.0752$ but that the magnitude of S is such that $0 \leq 1/S < \alpha_{m0}$. In such cases, the optimal WS consists of a cylindrical part and STE for which $\alpha_m = \alpha_{m0}(c_f)$ for all S from the indicated range, that is, it is again constant but of smaller magnitude. In the case of a circular base, one of the optimal WSs is the surface of a circular cylinder and the cone adjoining it with constant $\theta = \arcsin \alpha_{m0}(c_f) \geq \arcsin(1/S)$ which is represented in Fig. 1, *d*. The equality only holds in the limiting case when $S = 1/\alpha_{m0}(c_f)$. Two such WSs together with the "limiting" cone are shown in Fig. 4, *d*. The angle at the vertex of the identical cones in Fig. 4, *d* is smaller than the angle of the cone in Fig. 4, *a*. As S tends to infinity, the drag of the WS $\chi \to c_f S \to \infty$. As previously, cylindrical and conical parts can be combined in an arbitrary was while preserving the magnitudes of S and S_0 and $\alpha_m = \alpha_{m0}(c_f)$ in the conical segments. One of the optimal configurations which is obtained when this is done is shown in Fig. 4, *e*. Finally, as before, any cone in Fig. 4, *d* and *e* can be replaced by a spatial surface, constructed for $\alpha_m = \alpha_{m0}(c_f)$, of the type shown in Fig. 4, *a*.

Calculations, carried out for $c_f \leq c_{f*} \approx 1.0752$ naturally confirmed the superiority of a WS with end face SBE (when $\alpha_{m1} < 1/S < 1$) and cylindrical SBE (when $0 \leq 1/S < \alpha_{m0}$), although, for $c_f \ll 1$, the above-mentioned superiority is insignificant. As c_f increases, the reduction in drag becomes more noticeable first of all for bodies with end faces. The results of calculations for a friction coefficient $c_f = 0.5$, which is remarkable in that $\alpha_{m1}(0.5) = 1/\sqrt{2} \approx 0.7071$, and $\chi_m \equiv 1$ for all $1/\sqrt{2} \leq 1/S \leq 1$ give the curves $n = 4$ and $n = 2$ in Fig. 5. For $c_f = 0.5$ when $0 \leq 1/S < \alpha_{m0}(0.5) \approx 0.1265$, the optimal WSs contain cylindrical SBE. The magnitudes of $\Delta\chi = \chi/\chi_m - 1$, where $\chi = F(1/S)$ is the drag of a pointed cone which is "optimal" according to Refs. 2 and 5 are given for the above-mentioned range of values of $1/S$. The curve $n = 3$ gives the dependence of $\Delta\chi$ on $1/S$ for $c_f = c_{f*} \approx 1.0752$. Although, in this case, $\Delta\chi$ is approximately five times greater than for $c_f = 0.5$, the reduction in drag on changing to a combination of a cone and a cylinder constitutes tenths of a percent. For large $1/S$, the transition to truncated cones reduces the drag to an appreciably greater extent: up to 5% for $c_f = 0.5$ (curve *2* in Fig. 5) and up to 19% when $c_f = c_{f*}$ (the dashed line in Fig. 6).

For all of the WSs with STE considered above, where $0 < \alpha_m < 1$, just one of the necessary conditions for minimum drag (3.5) is violated when $c_f = c_{f*} \approx 1.0752$. This means that the optimal WS cannot have STE for such c_f. The unique possibility which remains in these cases is a cylinder with a blunt end face, that is, a surface formed by the SBE of the two types which are admitted by conditions (1.2). The transition from the optimal WSs with STE which have been constructed above occurs naturally as, when $c_f = c_{f*}$, their drag is equal to that of blunt cylinders with the same $0 \leq 1/S \leq 1$. In fact, $\alpha_{m0} = \alpha_{m1} = \alpha_{m*} \approx 0.2866$ in such cases and, by virtue of equalities (3.7) and (3.8), which are satisfied at the same time,

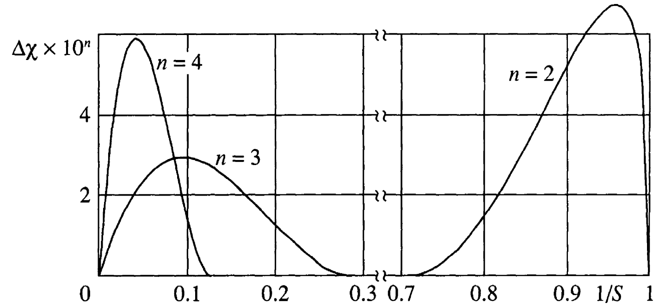

Fig. 5.

$$c_p(\alpha_{m*}) + \frac{c_f(\alpha_{m*})}{\alpha_{m*}}\sqrt{1 - \alpha_{m*}^2} = c_{p1} + c_{f0}\left(\frac{1}{\alpha_{m*}} - 1\right)$$

and the expressions obtained above for χ_m reduce to the equality

$$\chi_m = \Phi(\alpha_m) + Sc_{f0} = c_{p1} + c_{f0}(S - 1)$$

which holds for all $0 \leq 1/S \leq 1$ and not only in the "special" case considered earlier, corresponding to $1/S \approx 0.2866$ in the case of the LIM (1.4). The transition described is analogous to the transition to an end face discovered in the case of axisymmetric bodies[9] and spatial bodies,[10,11] where the radius of the base and the maximum admissible length of the body were fixed in the first case and just the area of the base in the second case.

For $c_f = c_{f*} \approx 1.0752$, the advantage of cylinders with a blunted end face over the "optimal" pointed cones rapidly increases as the friction coefficient becomes larger. The correctness of what has been said is confirmed in Fig. 6 for different friction coefficients c_f. The curve when $c_f = \infty$ was constructed using the formula

$$\Delta\chi = \sqrt{(S + 1)/(S - 1)} - 1$$

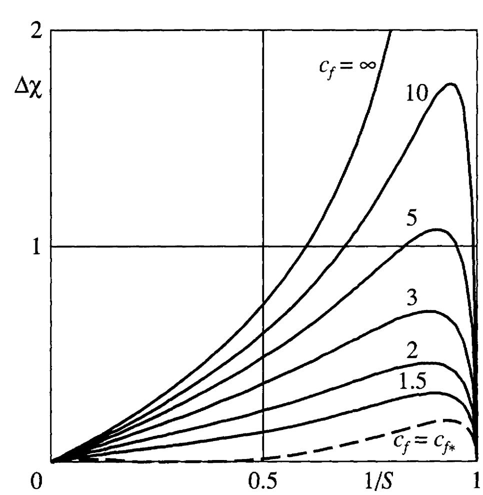

Fig. 6.

which holds when $0 \le 1/S < 1$. The introduction of end face and cylindrical segments in the case of values of c_f and $1/S$ for which the necessary conditions for minimum drag obtained above are violated leads to a decrease in drag compared with conical surfaces with $\alpha = 1/S$. Nevertheless, until now such segments have been called SBE only because the quantity α in them took limit values from the interval of applicability of the LIMs being considered which is determined by conditions (1.2).

We will now show that, in the corresponding cases, admissible modifications of end face or cylindrical surfaces can only lead to an increase in the drag.

An arbitrary admissible modification of the end face surface, optionally small, can lead to the introduction into it of a cone with $\alpha < 0$ as in Fig. 4, *c* when the area of the STEs for which $\alpha = \alpha_{m1}$ is reduced (retaining the magnitude of S). However, if this value of α differs from α_{m1}, then, unlike the initial WS, the resulting WS does not satisfy the necessary conditions for a minimum of χ. However, a modification with the introduction of a cone for which $\alpha = \alpha_{m1}$, is identical to the subdivision of the STE considered earlier (Fig. 4, *c*) which does not change the drag. Consequently, the modification which has been carried out does not reduce the drag. Analogous arguments hold for the cylindrical parts. The difference lies solely in the fact that, in the case of a cylindrical surface when $\alpha = 0$ is replaced by $\alpha > 0$, it is not possible, as it is in the case of an end face, to make direct use of non-optimal cones (now with $\alpha \ne \alpha_{m0}$). Without going into detail, we note that, in this situation, the motion ("rolling") along the modified surface of the base of an arbitrary circular cone with $\alpha > 0$ which has been described earlier in Ref. 12 can be used.

Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (05- 01- 00846) and within the framework of the Program for the Support of Leading Scientific Schools (NSh-9902.2006.1).

References

1. Kraiko AN, Pudovikov DE, Yakunina GE. *The Theory of Close to Optimal Aerodynamic Shapes*. Moscow: YaNUS-K; 2001.
2. Hull DG, Miele A. Three-dimensional shapes of minimum total drag. In: *Theory Optimum Aerodynamics Shapes.* N.Y.; L.: Acad. Press; 1965. pp. 319–338.
3. Berdichevskii VL. The shape of a body of minimum drag in a hypersonic gas flow. *Vestnik MGU Ser 1 Matematika Mekhanika* 1975;**3**:90–6.
4. Berdichevskii VL. *Variational Principles in Continaum Mechanics*. Moscow: Nauka; 2002.
5. Bunimovich AI, Dubinskii AV. *Mathematical Models and Methods of Localized Interaction Theory*. Singapore etc.: World Scient.; 1995. p. 226.
6. Tricomi FG. *Lezioni sulle Equazioni a Derivate Parziali*. Torino: Editice Gheroni; 1954.
7. Petrovskii GI. *Lectures on the Theory of Ordinary Differential Equations*. Moscow: Izd. MGU; 1984.
8. Yakunina GE. On optimal non-conical and asymmetric spatial configurations. *Prikl Mat Mekh* 2000;**64**(4):605–14.
9. Bryson AE. Effect of friction on optimum nonslender bodies of revolution. In: *Theory Optimum Aerodynamic Shapes*. New York: Acad. Press; 1965. pp. 275–282.
10. Yakunina GE. The construction of optimal spatial shapes using the localized interaction model. *Prikl Mat Mekh* 2000;**64**(2):299–310.
11. Yakunina GYe. Three-dimensional bodies of minimum total drag in hypersonic flow. *J Optimiz Theory and Appl* 2002;**115**(2):241–68.
12. Yakunina GE. Spatial shapes of minimum drag. *Vestnik Molodykh Uchenykh Ser Prikl Mat Mekh* 2000;**3**:77–85.

Translated by E.L.S.